



Hydrodynamic relation in a two-dimensional Heisenberg antiferromagnet in a field

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The spin stiffness ρ_s of a two-dimensional (2D) Heisenberg antiferromagnet depends nonanalytically on external magnetic field. We demonstrate that the hydrodynamic relation between ρ_s , the uniform susceptibility χ , and the spin-wave velocity c is not violated by such a behavior because similar nonanalytic terms from all three quantities mutually cancel out. In this work, explicit expressions for the field-dependent spin stiffness and for the magnon velocity of the 2D square-lattice antiferromagnet are obtained by direct calculation to order $1/S$ and in the whole range of magnetic fields.

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The effective description of spin waves in the Heisenberg and easy-plane antiferromagnets by a hydrodynamic theory goes back to the work by Halperin and Hohenberg.¹ Such a description implies the following hydrodynamic relation:

$$\frac{\chi c^2}{\rho_s} = 1, \quad (1)$$

between the susceptibility χ , the spin-wave velocity c , and the spin stiffness ρ_s . The importance of an independent verification of such a relation using direct microscopic calculations has been recognized and received a significant attention in the past.² Corresponding calculations confirming the validity of such a relation for the two-dimensional (2D) square-lattice Heisenberg antiferromagnet (HAF) have been carried out in the early 1990s using the spin-wave theory^{2,3} to orders $1/S^2$ and $1/S^3$. Numerical studies^{2,4,5} of the $S=1/2$ case of the same model have also given a strong support of the relation (1). While initial interest in this problem was motivated by the large- J high- T_c materials, more recently, synthesis of small- J quantum antiferromagnets⁶ has generated significant interest in the effects of external magnetic field in the properties of the HAFs, the regime that was previously unreachable.

Uniform magnetic field lowers the full rotational symmetry of the Heisenberg model to $O(2)$, making it equivalent to that of the easy-plane antiferromagnets with the easy plane of spin rotations perpendicular to the direction of the field. Note that the hydrodynamic consideration of Ref. 1 is also valid for the easy-plane antiferromagnets. Thus, at the first glance, it seems natural to assume that the two hydrodynamic descriptions should connect continuously. However, the situation is far less trivial as several quantities were shown to exhibit a nonanalytic behavior in small fields. In the earlier work,⁷ field dependence of the ground-state energy and susceptibility was discussed for the nonlinear σ model. The nonanalytic field-dependent corrections have been found in the dimensions $D \leq 3$. A subsequent independent study⁸ obtained the same nonanalytic behavior in small fields in the framework of the spin-wave theory. A recent

work⁹ used a hybrid $1/S$ -expansion- σ -model approach to demonstrate that the spin-wave velocity in a 2D antiferromagnet also has a nonanalytic dependence on the field $c(H) - c(0) \propto |H|$ in the first $1/S$ order. Recent studies of the combined effects of the Dzyaloshinskii-Moriya and uniform magnetic field in the spectrum and the ground-state properties^{10,11} of the 2D HAFs have also found nonanalytic dependencies that are related to the ones discussed here.

The origin of the nonanalytic behavior can be traced to the field-induced gap in one of the Goldstone modes. External field creates the so-called uniform-precession mode, which corresponds to the precession of the field-induced magnetization around the field direction with the energy equal to H . When the field is small, the mode is almost gapless and contributes to the fluctuation corrections to various quantities. These fluctuations may, potentially, induce nonhydrodynamic corrections in the corresponding $1/S$ order of the theory. Thus, the validity of the relation (1) in a field has to be verified.

In the case of the square-lattice HAF, out of three constants needed for the hydrodynamic relation, it is only the spin stiffness for which the presence or absence of the nonanalytic terms in the field dependence remains unknown. In this work we carry out direct analytical calculations of ρ_s to the necessary order in both $1/S$ and H to (i) identify such nonanalytic terms and (ii) verify that the nonanalytic behavior of all three quantities *does not* lead to the violation of the hydrodynamic relation (1). In the course of such derivation, we also obtain an analytic expression for the spin stiffness to order $1/S$ and for all ranges of the field. In addition, the nonanalytic behavior of the spin-wave velocity, previously obtained in Ref. 9 by a hybrid $1/S$ - σ -model approach, is confirmed within the framework of the spin-wave theory and the compact analytic expression for the velocity renormalization is obtained for an arbitrary value of the field.

We would like to make a separate note on the recent work¹² that combines a thorough numerical investigation of the static and dynamic properties of the $S=1/2$ square-lattice HAF in a field with the spin-wave analysis of the problem and provides a comprehensive comparison of the results.

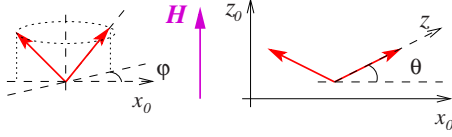


FIG. 1. (Color online) Left: angle of twist φ in the x_0 - y_0 plane of spins in one sublattice with respect to the other. Right: field-induced canting by the angle θ .

While in this work the spin stiffness is evaluated within the $1/S$ spin-wave approximation, it is done by numerical differentiation of the energy with respect to the twist angle and no nonanalytic behavior of ρ_s vs H is discussed. Also, the hydrodynamic relation is used to provide a better estimate of the spin-wave velocity within the $1/S$ approach, but the validity of it is not verified.¹²

We consider the spin- S HAF on the square lattice in an external field along the z_0 axis of the laboratory reference frame with the Hamiltonian given by

$$\hat{\mathcal{H}} = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - H \sum_i S_i^{z_0}, \quad (2)$$

where $\langle ij \rangle$ refer to the nearest-neighbor bonds. To study the spin stiffness, the Hamiltonian should be modified to introduce a twist angle between spins, rigidity, to which should yield the stiffness directly. One of the prescriptions⁴ is to twist spins in every second row by the fixed angle φ . Another, intuitively more symmetric approach is to twist all the spins in one sublattice relative to the other.¹² In the latter method the twist energy is 2 times larger than in the former case because every spin has twice as many nearest neighbors that are twisted. For the Heisenberg model on a bipartite lattice in zero field the direction of such a uniform twist is arbitrary. In the case of a nonzero external field, such a twist should be made in the plane perpendicular to the direction of the field, that is, in the x_0 - y_0 plane (see Fig. 1). Thus, using the sublattice twist with a small angle $\varphi \ll 1$, the modified Hamiltonian reads

$$\hat{\mathcal{H}} \approx J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - \frac{J\varphi^2}{2} \sum_{\langle ij \rangle} \mathbf{S}_i^\perp \cdot \mathbf{S}_j^\perp - H \sum_i S_i^{z_0}, \quad (3)$$

where $\mathbf{S}_i^\perp = (S_i^{x_0}, S_i^{y_0}, 0)$ and we have omitted the terms that are linear in φ , as they either vanish or contribute to the ρ_s term only in the higher ($1/S^2$) order.^{2,3} As such, the Hamiltonian (3) contains all the necessary terms to study both the classical limit of the model and the $1/S$ fluctuation corrections to it.

To study quantum fluctuations around the classical spin configuration it is convenient to transform spins to ‘‘rotating’’ local reference frames in which the quantization axis z is along the classical spin direction.^{8,13,14} Magnetic field cants spins toward its direction as is shown in Fig. 1. Assuming that the spins lie in the x - z plane we perform transformation from the laboratory frame (x_0, z_0) into the rotating frame (x, z) ,^{8,13}

$$S_i^{z_0} = S_i^z \sin \theta - e^{i\mathbf{Q}r_i} S_i^x \cos \theta,$$

$$S_i^{x_0} = e^{i\mathbf{Q}r_i} S_i^z \cos \theta + S_i^x \sin \theta, \quad S_i^{y_0} = S_i^y, \quad (4)$$

where $\mathbf{Q} = (\pi, \pi)$ is the ordering wave vector and canting angle is as shown in Fig. 1. The spin Hamiltonian (3) in the local coordinate system (4) takes the form

$$\begin{aligned} \hat{\mathcal{H}} \approx J \sum_{\langle ij \rangle} & \left[S_i^y S_j^y - \cos 2\theta (S_i^z S_j^z + S_i^x S_j^x) \right. \\ & \left. - \frac{\varphi^2}{2} (S_i^y S_j^y + S_i^x S_j^x \sin^2 \theta - S_i^z S_j^z \cos^2 \theta) \right], \\ & - H \sin \theta \sum_i S_i^z, \end{aligned} \quad (5)$$

where, again, the terms that are not contributing to the harmonic approximation are omitted.

At the first glance, the spin stiffness can be defined from the averaging of the second line in Eq. (5) over the ground state. However, the situation is slightly more complex as the field-induced canting angle θ , which should be found from the minimization of the classical energy in Eq. (5),⁸ also depends on the twist angle.¹² Performing such a minimization for Eq. (5),⁸ one obtains

$$\sin \theta = h \left(1 + \frac{\varphi^2}{4} \right), \quad (6)$$

where the terms of higher order in the twist angle are truncated and the dimensionless variable $h = H/(8JS)$, the field normalized to the saturation field $H_s = 8JS$ at which spins become fully aligned, is introduced. With the help of Eq. (6) one can eliminate θ in Eq. (5) to obtain

$$\hat{\mathcal{H}} \approx \hat{\mathcal{H}}_{\varphi=0} + \hat{\mathcal{H}}_{\rho_s}, \quad (7)$$

where $\hat{\mathcal{H}}_{\varphi=0}$ contains no twist angle and $\hat{\mathcal{H}}_{\rho_s}$ is given by

$$\hat{\mathcal{H}}_{\rho_s} = \frac{J\varphi^2}{2} \left\{ \sum_{\langle ij \rangle} [(1+h^2)S_i^z S_j^z + h^2 S_i^x S_j^x - S_i^y S_j^y] - 4Sh^2 \sum_i S_i^z \right\}. \quad (8)$$

The subsequent treatment of the Hamiltonian $\hat{\mathcal{H}}_{\varphi=0}$ involves standard bosonization of spin operators via the Holstein-Primakoff transformation to the first $1/S$ order

$$S_i^z = S - a_i^\dagger a_i, \quad S_i^- \approx a_i^\dagger \sqrt{2S}, \quad S_i^+ = (S_i^-)^\dagger, \quad (9)$$

which is followed by the Bogolyubov transformation.⁸ This yields the linear spin-wave theory Hamiltonian

$$\hat{\mathcal{H}}_{\varphi=0} \approx E_{\text{GS}} + 4JS \sum_{\mathbf{k}} \omega_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} \quad (10)$$

(see Ref. 8 for details). The dimensionless frequency is

$$\omega_{\mathbf{k}} = \sqrt{(1 + \gamma_{\mathbf{k}})[1 - (1 - 2h^2)\gamma_{\mathbf{k}}]} \quad (11)$$

and $\gamma_{\mathbf{k}} = (\cos k_x + \cos k_y)/2$.

After diagonalization of $\hat{\mathcal{H}}_{\varphi=0}$, the spin stiffness can be found as a coefficient in front of φ^2 in the twist part $\hat{\mathcal{H}}_{\rho_s}$ of the Hamiltonian (8) by averaging the spin operators over the

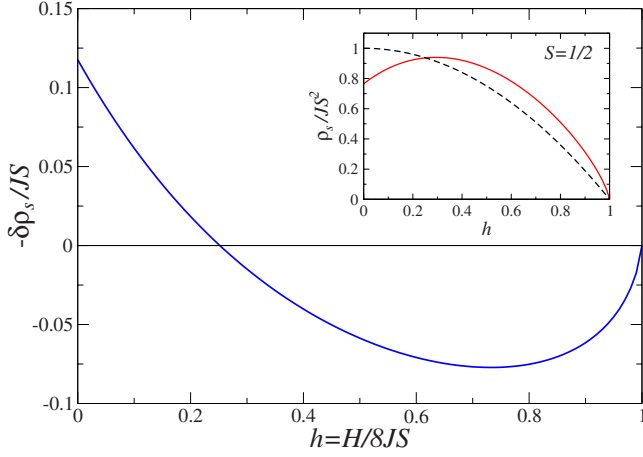


FIG. 2. (Color online) $1/S$ quantum correction to the spin stiffness $-\delta\rho_s/JS$ as a function of the field h . Inset: ρ_s/JS^2 vs h for $S=1/2$; dashed line is ρ_s^{cl}/JS^2 vs h .

spin-wave ground state and keeping terms to $1/S$ order. Having in mind the extra factor of 2 in the sublattice twist approach, this finally yields

$$\delta E_\varphi = \langle \hat{\mathcal{H}}_{\rho_s} \rangle = \varphi^2 N \rho_s, \quad (12)$$

where $\rho_s = \rho_s^{\text{cl}} + \delta\rho_s$, with the classical and quantum contributions given by

$$\rho_s^{\text{cl}} = JS^2(1 - h^2),$$

$$\delta\rho_s = -JS[2n - \Delta(1 + h^2) + m(1 - h^2)], \quad (13)$$

where we use the following Hartree-Fock averages of the two-boson operator combinations:

$$\begin{aligned} n &= \langle a_i^\dagger a_i \rangle = \frac{1}{2} \sum_{\mathbf{q}} \left[\frac{1 + h^2 \gamma_{\mathbf{q}}}{\omega_{\mathbf{q}}} - 1 \right], \\ \Delta &= \langle a_i a_j \rangle = \frac{1}{2} (1 - h^2) \sum_{\mathbf{q}} \frac{\gamma_{\mathbf{q}}^2}{\omega_{\mathbf{q}}}, \\ m &= \langle a_i^\dagger a_j \rangle = \frac{1}{2} \sum_{\mathbf{q}} \frac{\gamma_{\mathbf{q}} + h^2 \gamma_{\mathbf{q}}^2}{\omega_{\mathbf{q}}}. \end{aligned} \quad (14)$$

The above result (13) gives ρ_s to the order $1/S$ and for the fields anywhere between zero and the saturation value. At $H=0$ the expression for ρ_s in Eqs. (13) and (14) coincides with the known zero-field formula for the Heisenberg model.^{2,3} The field dependence of the quantum correction to the spin stiffness $\delta\rho_s$ is shown in Fig. 2. The inset presents ρ_s for the spin-1/2 case. One of the interesting observations is that $\delta\rho_s$ changes sign as a function of the field. It also exhibits a singular behavior in the derivative as $h \rightarrow 1$, similar to the one discussed before for the magnetization,⁸ and is related to the logarithmically vanishing scattering amplitude in the dilute 2D gas of bosons. The linear (nonanalytic) field dependence at small field is also clear from Fig. 2.

With the expressions (13) and (14) at hand, one can now study the field dependence of ρ_s at $h \rightarrow 0$. After some algebra one finds

$$\rho_s = \rho_s^{H=0} - JS \sum_{\mathbf{k}} \left(\frac{1}{\omega_{\mathbf{k}}} - \frac{1}{\omega_{\mathbf{k}}^{H=0}} \right) + \mathcal{O}(h^2). \quad (15)$$

It is easy to see that due to the field-induced gap $\propto H$ in the magnon spectrum the fluctuation terms such as the one in Eq. (15) are yielding corrections $\propto |H|$ in 2D. Some further algebra gives

$$\rho_s = JS^2 \left(Z_{\rho_s} + \frac{2}{\pi S} |h| \right) + \mathcal{O}(h^2, 1/S^2) \approx \rho_s^{H=0} \left(1 + \frac{2}{\pi S} |h| \right), \quad (16)$$

where $\rho_s^{H=0}$ contains zero-field $1/S$ renormalization factor Z_{ρ_s} (Ref. 2) and the last expression is obtained within the same $1/S$ accuracy.

For completeness, we also list the corresponding $1/S$ expressions for the susceptibility. Magnetization of the square-lattice HAF is $M = M_{\text{cl}} + \delta M$ with the classical part and the quantum correction given by⁸

$$M_{\text{cl}} = Sh, \quad \delta M = -h(m + \Delta). \quad (17)$$

Using $\chi = \partial M / \partial H$ yields

$$\chi = \chi_{\text{cl}} + \delta\chi = \frac{1}{8J} \left[1 - \frac{1}{S} (m + \Delta - h^2 I_1) \right], \quad (18)$$

where I_1 stands for the integral

$$I_1 = \sum_{\mathbf{q}} \frac{\gamma_{\mathbf{q}}^2 (1 + \gamma_{\mathbf{q}})^2}{\omega_{\mathbf{q}}^3}. \quad (19)$$

At small fields, the same algebra as above gives

$$\chi = \frac{1}{8J} \left(Z_\chi + \frac{4}{\pi S} |h| \right) + \mathcal{O}(h^2, 1/S^2) \approx \chi^{H=0} \left(1 + \frac{4}{\pi S} |h| \right). \quad (20)$$

Figure 3 shows the field dependence of the quantum correction to the susceptibility $\delta\chi$. The inset presents χ for the case of $S=1/2$. Similarly to $\delta\rho_s$, $\delta\chi$ changes sign as a function of the field. As is discussed in Ref. 8, χ has a singular logarithmic behavior at $h \rightarrow 1$ and the linear nonanalytic field dependence at $h \rightarrow 0$ is also clearly seen in Fig. 3.

For the square-lattice HAF in external magnetic field the energy of magnons to the first order of the $1/S$ expansion is given by¹³

$$\bar{\epsilon}_{\mathbf{k}} = 4JS\omega_{\mathbf{k}} + \delta\epsilon_{\mathbf{k}}^{(1)} + \delta\epsilon_{\mathbf{k}}^{(2)}, \quad (21)$$

where $\omega_{\mathbf{k}}$ is defined in Eq. (11), correction $\delta\epsilon_{\mathbf{k}}^{(1)}$ includes the Hartree-Fock, and the canting angle renormalizations

$$\begin{aligned} \delta\epsilon_{\mathbf{k}}^{(1)} &= \frac{4J}{\omega_{\mathbf{k}}} \{ \Delta - n + h^2(\Delta + m)[1 - 2\gamma_{\mathbf{k}}(1 - h^2)] \\ &\quad + \gamma_{\mathbf{k}}^2 [n - \Delta(1 - h^2 + 2h^4) - mh^2(3 - 2h^2)] \} \end{aligned} \quad (22)$$

and $\delta\epsilon_{\mathbf{k}}^{(2)}$ is the one-loop contributions from the three-magnon coupling

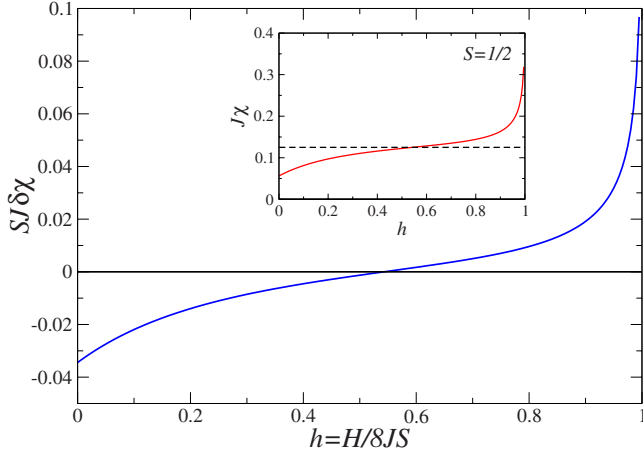


FIG. 3. (Color online) $1/S$ quantum correction to the susceptibility $J\delta\chi$ as a function of the field h . Inset: $J\chi$ vs h for $S=1/2$; dashed line is $J\chi_{cl}$.

$$\delta\varepsilon_{\mathbf{k}}^{(2)} = -4Jh^2(1-h^2) \sum_{\mathbf{q}} \left[\frac{\tilde{\Gamma}_1(\mathbf{k}, \mathbf{q})^2}{\omega_{\mathbf{q}} + \omega_{\mathbf{k}-\mathbf{q}+\mathbf{Q}} - \omega_{\mathbf{k}}} + \frac{\tilde{\Gamma}_2(\mathbf{k}, \mathbf{q})^2}{\omega_{\mathbf{k}} + \omega_{\mathbf{q}} + \omega_{\mathbf{k}+\mathbf{q}-\mathbf{Q}}} \right]. \quad (23)$$

Explicit expressions for $\tilde{\Gamma}_1(\mathbf{k}, \mathbf{q})$ and $\tilde{\Gamma}_2(\mathbf{k}, \mathbf{q})$ are given in Ref. 13. After some algebra, the $1/S$ correction to the spin-wave velocity can be written as

$$\frac{c - c_0}{c_0} = \frac{\Delta(1 - h^2 + h^4) - n + mh^2(2 - h^2)}{S(1 - h^2)} - \frac{h^2}{2S} I_1 - \frac{2h^2}{S} \left(\frac{I_2(\mathbf{k}) - 4\Delta}{k^2} \right) \Bigg|_{k \rightarrow 0}, \quad (24)$$

where the bare spin-wave velocity is $c_0^2 = 8J^2S^2(1-h^2)$ and the \mathbf{k} -dependent function in the last term is

$$I_2(\mathbf{k}) = 2(1 - h^2) \sum_{\mathbf{q}} \gamma_1 \frac{\gamma_1 \alpha_1 \beta_2 + \gamma_2 \omega_1 \omega_2}{\omega_1 \omega_2 (\omega_1 + \omega_2)}, \quad (25)$$

with $\alpha_1 = (1 + \gamma_1)$, $\beta_2 = [1 - (1 - 2h^2)\gamma_2]$, and $1, 2 = \mathbf{q}, \mathbf{q} - \mathbf{k}$.

The field dependence of the quantum correction to the spin-wave velocity Δc , obtained by numerical evaluation of the integrals in Eq. (24), is shown in Fig. 4. The inset presents the normalized magnon velocity for $S=1/2$. The linear (nonanalytic) field dependence at small field is clearly visible in Fig. 4. Behavior of Δc at $h \rightarrow 1$ is also singular, similarly to other quantities. It is interesting to note that the correction to the spin-wave velocity Δc is almost flat for $0.2 \leq h \leq 0.9$.

It is easy to see that the first Hartree-Fock term in Eq. (24) does not contribute to the anomalous nonanalytic field dependence. After some more algebra, one can show that the same is true for the last term in Eq. (24). The second term in Eq. (24), on the other hand, yields at $h \rightarrow 0$,

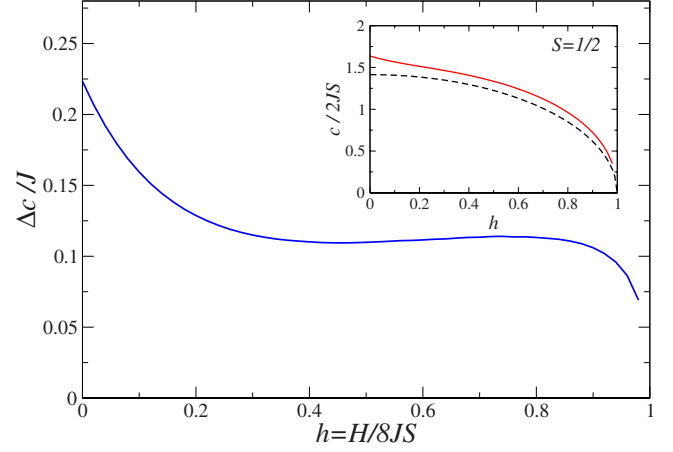


FIG. 4. (Color online) $1/S$ quantum correction to the spin-wave velocity $\Delta c/J$ as a function of the field h . Inset: $c/2JS$ vs h for $S=1/2$; dashed line is $c_0/2JS$ vs h .

$$\frac{\Delta c}{c_0} \approx -\frac{2h^2}{S} \sum_{\mathbf{q}} \left(\frac{1}{(\omega_{\mathbf{q}})^3} - \frac{1}{(\omega_{\mathbf{q}}^{H=0})^3} \right) = -\frac{|h|}{\pi S}. \quad (26)$$

This gives the same result as in Ref. 9,

$$c^2 \approx c_{H=0}^2 \left(1 - \frac{2}{\pi S} |h| \right). \quad (27)$$

Combining the expressions for the small-field expansion of all three quantities ρ_s , χ , and c from Eqs. (16), (20), and (27), one can easily see that the hydrodynamic relation (1) is obeyed as all the nonanalytic terms explicitly cancel each other in the leading order in h . Moreover, such a verification of the relation (1) can be extended to an arbitrary field. Expanding the hydrodynamic relation (1) to $1/S$ order and observing that it is fulfilled at the classical level, $\chi_{cl} c_0^2 / \rho_s^{cl} = 1$, one concludes that for the hydrodynamic relation to exist the

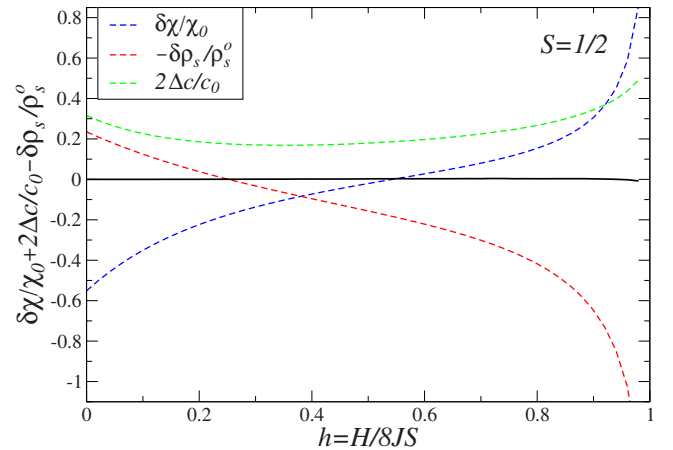


FIG. 5. (Color online) Cancellation of the $1/S$ quantum corrections in the hydrodynamic relation (28) as a function of the field h (solid line). Dashed lines show contributions of individual terms in Eq. (28).

quantum corrections from all three quantities must cancel each other at any field in each order of $1/S$. For the $1/S$ corrections this leads to

$$\frac{\chi c^2}{\rho_s} - \frac{\chi_{cl} c_0^2}{\rho_s^{cl}} \approx \left(\frac{\delta\chi}{\chi_{cl}} + 2 \frac{\Delta c}{c_0} - \frac{\delta\rho_s}{\rho_s^{cl}} \right) = 0. \quad (28)$$

Numerical verification of the above relation is made using expressions (13), (18), and (24) and is presented by the solid line in Fig. 5. The dashed lines show contributions of individual terms in Eq. (28). One can conclude that the cancellation takes place for all values of $0 < h < 1$.

Having in mind the relation (28), we can now obtain a much simpler expression for the spin-wave velocity renormalization in magnetic field

$$\frac{c - c_0}{c_0} = \frac{1}{S} \left(\frac{\Delta - n}{1 - h^2} - \frac{h^2}{2} I_1 \right), \quad (29)$$

where I_1 is defined in Eq. (19).

Altogether, we have confirmed the validity of the hydro-

dynamic relation for the 2D Heisenberg antiferromagnet in a uniform field. Despite the appearance of the nonanalytic terms in the field dependence of all key quantities due to quantum fluctuation involving small field-induced gap, they are not sufficient to violate such a relation. We have obtained expressions for the spin stiffness ρ_s and for the spin-wave velocity c for the square-lattice HAF, valid to the first order in $1/S$ and for the whole range of magnetic fields. The nonanalytic field dependence of c , previously obtained by a hybrid $1/S$ -expansion $-\sigma$ -model approach, is verified using the more conventional spin-wave theory.

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